

Title	Asymptotic Completeness for Hamiltonians with Time-dependent Electric Fields(Spectral and Scattering Theory and Its Related Topics)
Author(s)	Yokoyama, Koichiro
Citation	数理解析研究所講究録 (1997), 994: 72-75
Issue Date	1997-05
URL	<a href="http://hdl.handle.net/2433/61204">http://hdl.handle.net/2433/61204</a>
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

# Asymptotic Completeness for Hamiltonians with Time-dependent Electric Fields

大阪大学理学研究科 横山 耕一郎 (Koichiro Yokoyama)

## 1 Introduction

We consider the following equation,

$$i\partial_t u(t, x) = H(t)u(t, x) \quad \mathbf{H} = \mathbf{L}^2(\mathbf{R}^\nu), \quad (1)$$

$$H(t) = -\frac{1}{2}\Delta - E(t) \cdot x + V(x) \quad (\nu \geq 1)$$

with  $E(t) = E + e(t)$ ,  $E$  being a nonzero constant vector in  $\mathbf{R}^\nu$ .

We assume  $V(x)$  is real valued and short range (i.e.  $V(x) = O(|x|^{-1/2-\epsilon})$   $|x| \rightarrow \infty$ ). As is well-known, with some suitable conditions on  $V(x)$  and  $E(t)$ ,  $H(t)$  generates a unique unitary propagator  $\{U(t, s)\}_{-\infty < t, s < \infty}$ . We denote the unitary propagator generated by  $H_0(t) = H(t) - V(x)$  as  $\{U_0(t, s)\}$ .

Studies for Schrödinger operators with electric fields have been done mainly for D.C. and A.C. Stark effects. Asymptotic completeness for A.C. Stark Hamiltonian, which is represented by  $E(t) \cdot x = (\cos t)x_1$ , was first proved by Howland and Yajima in [How] and [Ya]. In these papers they consider operators  $K = -i\frac{d}{dt} + H(t)$  and  $K_0 = K - V$  on  $L^2(\mathbf{T} \times \mathbf{R}^\nu)$  and prove the asymptotic completeness by reducing it to that for  $K$  and  $K_0$ . These results were extended to the 3-body case by Nakamura [Na]. The asymptotic completeness of modified wave operator for long-range potential was proved by Kitada-Yajima [K-Y]. Recently asymptotic completeness for  $E(t) = E + (\cos t)\mu$  by Møller [Mø] ( $\mu$  is small enough compared with the main field  $E$ )

As for the case  $E(t) = E$ , the asymptotic completeness for long-range many-particle systems was proved by Adachi and Tamura in [AT1] [AT2]. In these papers they show the propagation estimates for the propagator by using the commutator technique of E.Mourre [Mo].

The aim of this paper is to accomodate the propagation estimates for the constant electric fields to the Schrödinger operator of the form (1) allowing  $e(t)$  to be nonperiodic but small as  $t \rightarrow \infty$ . And with these results, we prove the existence and asymptotic completeness of wave operators.

We assume that  $V(x) \in C^\infty(\mathbf{R}^\nu)$  and there exists  $\delta_0 > 1/2$  such that

$$|\partial_x^\alpha V(x)| \leq C_\alpha \langle x \rangle^{-\delta_0 - |\alpha|} \quad \forall \alpha \quad (2)$$

where  $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$ .

In this paper, either of the following two assumptions are imposed on  $V(x)$  and  $e(t)$ . The former requires that  $V(x)$  is relatively small for  $|E|$ . And the latter requires  $|e(t)| \rightarrow 0$  as  $t \rightarrow \infty$ .

**Assumption 1** *We assume*

$$|E| > \sup_{x \in \mathbf{R}^\nu} \frac{E}{|E|} \cdot \nabla_x V(x). \quad (3)$$

*There exist  $c(t) \in C^2(\mathbf{R})$  and  $\eta_0 > 0$  satisfying*

$$|\dot{c}(t)| = O(t^{-\eta_0}) \quad t \rightarrow \infty, \quad (4)$$

$$\ddot{c}(t) = -e(t). \quad (5)$$

With this Assumption we write

$$b(t) = -\dot{c}(t), \quad (6)$$

$$a(t) = \frac{1}{2} \int_0^t |\dot{c}(\theta)|^2 d\theta. \quad (7)$$

**Assumption 2**  *$e(t)$  is a continuous integrable function on  $\mathbf{R}_+$ . Let  $b(t)$  be defined by*

$$b(t) = - \int_t^\infty e(s) ds. \quad (8)$$

*Then  $b(t)$  satisfies*

$$E \cdot b(t) \equiv 0 \quad t \gg 1, \quad (9)$$

*and there exists  $u_0 > 5/2$  such that  $|b(t)| = O(t^{-u_0})$*

Under this Assumption we put

$$c(t) = \int_t^\infty b(s) ds, \quad a(t) = -\frac{1}{2} \int_t^\infty |b(s)|^2 ds. \quad (10)$$

On each of these Assumptions 1 or 2,  $H(t)$  is essentially self-adjoint on  $D(|x|) \cap H^2(\mathbf{R}^\nu)$ . And we can construct unique unitary propagator satisfying the following properties (see [Ya2].)

For all  $t, t', s \in \mathbf{R}$ ,

$$U(t, t) = I, \quad U(t, s)U(s, t') = U(t, t'), \quad (11)$$

$$\frac{d}{dt}U(t, s) = -iH(t)U(t, s). \quad (12)$$

We also denote the unitary propagator associated with  $H_0(t)$  as  $U_0(t, s)$ . Our main result is the following.

**Theorem 3** *Suppose Assumption 1 or 2 holds. Then the following strong limit exist.*

$$W^+(s) = s - \lim_{t \rightarrow +\infty} U_0(t, s)^* U(t, s) \quad (13)$$

$$\tilde{W}^+(s) = s - \lim_{t \rightarrow +\infty} U(t, s)^* U_0(t, s) \quad (14)$$

**Remark 4** *Theorem 3 holds as  $t \rightarrow -\infty$ , if we replace  $\infty$  in Assumption 1 and 2 by  $-\infty$ .*

## 2 Translated Hamiltonians

At first we introduce a Hamiltonian  $\hat{H}(t)$ , which is obtained by translating  $H(t)$ . In this section, we give the propagation estimates for the propagator  $\hat{U}(t, s)$  associated with  $\hat{H}(t)$ .

**Definition 5**

$$\hat{H}(t) = -\frac{1}{2}\Delta - E \cdot x + V(x - c(t)) + E \cdot c(t). \quad (15)$$

We also denote  $\hat{H}(t) - V(x - c(t))$  as  $\hat{H}_0(t)$ .

We can also construct a unique unitary propagator  $\hat{U}(t, s)$  and  $\hat{U}_0(t, s)$ , generated by  $\hat{H}(t)$  and  $\hat{H}_0(t)$ . We remark that  $U(t, s)$  and  $\hat{U}(t, s)$  ( $U_0(t, s)$  and  $\hat{U}_0(t, s)$ ) are related through the following relation.

**(Avron-Herbst formula)**

$$U(t, s) = \tau(t)\hat{U}(t, s)\tau^*(s), \quad (16)$$

where

$$\tau(t) = \exp(ia(t)) \exp(-ib(t) \cdot x) \exp(ic(t) \cdot p) \quad , \quad p = -i\nabla_x. \quad (17)$$

**Theorem 6** *We assume Assumption 1. Then there exists  $\sigma > 0$  such that for all  $0 < u \leq 2$  and  $h \in C_0^\infty(\mathbf{R})$*

$$\|F(\frac{|x|}{t^2} \leq \sigma)\hat{U}(t, s)h(\hat{H}(s)) < x >^{-u/2} \|_{B(\mathbf{H})} = O(t^{-L}) \quad (t \rightarrow \infty), \quad (18)$$

with  $L = \min\{u, 3/2, 1 + \eta_0\}$ .

**Theorem 7** *We assume Assumption 2. Then there exists  $\sigma > 0$  such that for all  $0 < u \leq \min\{u_0/2, 3/2\}$  and  $f \in C_0^\infty(\mathbf{R})$*

$$\|F(\frac{|x|}{t^2} \leq \sigma) f(\hat{H}(t)) \hat{U}(t, s) h(\hat{H}(s)) < x >^{-u/2} \| = O(t^{-L}) \quad (t \rightarrow \infty) \quad (19)$$

where  $L = \min\{u_0, 3/2\}$ .

**Remark 8** *Theorem 3 is obtained if we show the existence of the strong limits of  $\hat{U}_0(t, s)^* \hat{U}(t, s)$  and  $\hat{U}(t, s)^* \hat{U}_0(t, s)$ . We can prove them by using Cook's method and Theorem 6 (Theorem 7).*

## References

- [AT1] Adachi, T. Tamura, H. :Asymptotic completeness for long-range many-particle systems with Stark Effect.  
J.Math.Sci.,The Univ. of Tokyo
- [AT2] Adachi, T. Tamura, H. :Asymptotic completeness for long-range many-particle systems with Stark Effect,II.  
Commun.Math.Phys.
- [How] Howland, J. :Scattering Theory for Hamiltonians Periodic in Time.  
Indiana Univ.Math.Journal**28**,471–494(1979)
- [K-Y] Kitada, H. Yajima, K. :A scattering theory for time-dependent long range potentials. Duke Math.J.**49**,341–376(1982)
- [Mø] Møller, J. S. :Two-body short-range systems in a periodic electric field.  
Tokyo University Preprint, (1996)
- [Mo] Mourre, E. :Absence of singular continuous spectrum for certain self-adjoint operators. Commun.Math.Phys.**78**,391–408(1980)
- [Na] Nakamura, S. :Asymptotic completeness for three-body Schrödinger equations with time periodic potentials.  
J.Fac.Sci.Univ.Tokyo Sect.IA,Math.**33**,379–402(1986)
- [Ya] Yajima, K. :Scattering theory for Schrödinger operators with potentials periodic in time. J.Math.Soc.Jpn.**29**,729–743(1977)
- [Ya2] Yajima, K. :Existence of solutions for Schrödinger evolution equations.  
Commun.Math.Phys.**110**,415–426(1987)